4.1 Folding Rules

Due to their small thickness, membranes can be easily bent, but are comparatively difficult to stretch. Hence, in studying the packaging of membranes it is normal to model them as inextensional plates of zero thickness.

Inextensional deformation can be linked, in general, to an intrinsic property of surfaces, the Gaussian curvature. Following Calladine (1983) the in-plane deformation of a flat plate or shallow shell can be linked to its change of Gaussian curvature through the equation

$$dK = -\frac{\partial^2 \epsilon_{yy}}{\partial x^2} + \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} - \frac{\partial^2 \epsilon_{xx}}{\partial y^2}$$ (4.1)

For inextensional deformation $\epsilon_{xx} = \epsilon_{yy} = \gamma_{xy} = 0$ and hence

$$dK = 0$$

Therefore, during packaging an initially flat membrane, whose Gaussian curvature is zero everywhere, is only allowed to deform in such a way that its Gaussian curvature does not change.

Packaging schemes for thin membranes make extensive use of straight, localised creases and it will now be shown that some general conditions have to be satisfied when two or more creases intersect at a point.

1 Based, with the exception of Section 4.3, on a chapter of the forthcoming book “Structural Concepts” by K. Miura and S. Pellegrino.

2 Current address: Department of Mechanical Engineering, The University of Bath, U.K.
Because a given amount of Gaussian curvature is geometrically represented by an area over a sphere of unit radius, zero Gaussian curvature is associated with a zero area on this sphere. For the case that we are studying this means that the area on the unit sphere associated with any change of configuration of the membrane during packaging should be represented by a zero area.

This observation has several important practical implications. Consider the flat membrane shown in Fig. 4.1(a), which is divided into two parts by a single, straight crease. The image on the unit sphere of the patch drawn on the membrane is the single point shown, and hence the area on the unit sphere associated with this patch is \( A = 0 \).

![Figure 4.1. Flat membrane with a single fold, and its spherical image.](image)

Now, fold down part 2 of the membrane, through \( 90^\circ \). The spherical image of part 1 is unchanged, whereas the image of part 2 moves along an arc of a great circle, as shown in Fig. 4.1(b). The change of Gaussian curvature of the membrane during this operation is zero, because the area of this circular arc is, obviously, zero.

Next, consider the membrane shown in Fig. 4.2(a), which is divided into three parts by three straight creases that meet at a common point. The spherical image of any chosen patch on the membrane is, again, a single point, but this time if part 2 of the membrane is folded down through \( 90^\circ \) without moving part 1 —this motion fully defines the direction of the normal vector \( n_3 \)— the spherical image of the chosen patch has non-zero area. This implies that the Gaussian curvature has to increase, i.e. part 3 of the membrane has to stretch.

Finally, consider the membrane shown in Fig. 4.3(a), divided into four parts by four straight folds with a common point. As in the previous cases, let us fold part 2 down by rotating the first crease through \( 90^\circ \). It is now possible to determine the magnitude by which the crease between parts 3 and 4 should be rotated so that all the parts fit together, as shown in Fig. 4.3(b). Now consider the spherical image of the patch marked on the membrane: it is a skew quadrangle whose
Figure 4.2. Flat membrane with three folds, and its spherical image.

edges are arcs of great circles. To calculate its area we have to be careful, though, because when we go from point 1 to point 2 and then to 3 on the sphere we are following a curve in an anti-clockwise sense, but when we go from point 3 to point 4 and then to point 1 we are turning in the opposite sense. Therefore, the area enclosed by the curve 1234 is given by

\[ \Delta A = A_1 + A_2 \]

but \( A_1 = -A_2 \) and hence \( \Delta A = 0 \). Note that the way in which the zero area-change condition has been satisfied is by forming three hill, or convex folds and one valley, or concave fold. Of course, it would have also been possible to form three valley folds and one hill fold. We introduce the fold sign convention that hill folds are defined to be positive.

In conclusion, inextensional folding of a membrane requires that whenever different creases meet at a common point there should be at least four folds, of which three have one sign, and one fold has the opposite sign.

Having explored the conditions that are imposed by the inextensionality constraint on the design of the folding pattern, an additional condition is imposed by the requirement for compact packaging. In simple terms, it is required that there should be no voids in the package. For example, in the case of four folds meeting at a common point \( O \), Fig. 4.4, to avoid any voids the difference between two adjacent angles has to be equal to the difference between the remaining two angles

\[ \alpha_4 - \alpha_1 = \alpha_3 - \alpha_2 \]

Re-arranging, we obtain

\[ \alpha_1 + \alpha_3 = \alpha_2 + \alpha_4 \]

In Origami this condition, Eq. 4.2, is known as Hushimi’s theorem.
Figure 4.3. Flat membrane with four folds, and its spherical image.

Figure 4.4. Packaging without voids.
4.2 Miura-ori

In this section we investigate schemes for folding a rectangular membrane into a package of flat shape. There is a standard way of doing this, which might be called “letter folding”, where a rectangular sheet is folded about its centre line, the folded sheet thus obtained is then folded about its own centre line, and the same operation is repeated, say, five times. If one opens the sheet, one finds the folding pattern shown in Fig. 4.5(a). Note that: (i) the three-to-one condition on the sign of the folds meeting at the 21 points of intersection of the folds, Section 4.1, is satisfied and (ii) the arrangement of hill and valley folds appears fairly random.

![Figure 4.5](image.png)

Figure 4.5. (a) Letter folding; (b) map folding.

Also, an examination of the unfolding of a sheet packaged according to this technique shows that it would be a very complex process to carry out automatically. The problem is that the unfolding process involves a multi-step sequence where motion of selected folds starts and then stops.

An alternative to this way of packaging a thin membrane, Fig. 4.5(b), is the folding scheme commonly used for road maps. To fold a sheet of paper according to this scheme, which is the reader is invited to try, one folds the sheet first into a concertina, thus producing a strip with a width of $1/8$ of the original width of the sheet and its full height. Then, this strip is folded again into a concertina. Figure 4.5(b) shows that the new scheme satisfies, again, the three-to-one sign rule, but is much more regular.

Now the unfolding process is much simpler, as the sheet can be deployed completely by pulling apart the two corners of the sheet that are right at the top and the bottom of the package.
The unfolding process involves the sheet opening first in one direction and then, suddenly, in the other direction, i.e. it mirrors exactly the folding process.

Figure 4.6 shows a folding pattern —known as Miura-ori— which was devised by Miura (1980), where the vertical folds are no longer straight but arranged along zig-zag lines at angles of $\pm \alpha$ to the vertical. This may seem only a small difference from the pattern of Fig. 4.5(b), but it introduces a radical change in the unfolding behaviour.

Figure 4.6. Miura-ori. The dimensions given, in millimetres, are for a model made from an A2 sheet. A model can be made also from an A4 sheet, but all dimensions should be divided by 2.

The unfolding sequence of a sheet packaged according to this scheme is shown in Fig. 4.7: the expansion is biaxial and hence, if the sheet is unfolded by pulling two opposite corners apart, it gradually expands both in the direction of pulling and in the direction orthogonal to it. Changing the fold angles in Fig. 4.5(b) from $90^\circ$ to $90^\circ \pm 6^\circ$ has had the effect of coupling the motions in the two directions.

A fundamental difference between the unfolding behaviour of a sheet packaged according to the road-map scheme and Miura-ori is that the first deploys in a kind of sequential fashion, whereas the second deploys synchronously. If we imagine mechanical models of these two folding schemes, made from thin plates connected by piano hinges, the Miura-ori scheme leads to a system with a single degree of freedom during deployment, whereas the other scheme has many degrees of freedom in any configuration.

Another advantage of Miura-ori is that nested folds are offset by a small amount, which has the effect of reducing the maximum curvature of the membrane, thus reducing the stress level associated with creasing.

Next, we discuss some geometric details of this packaging scheme. First, let us consider the envelope of the packaged membrane. From Fig. 4.6(b) the largest dimension is $h$ and, to keep $h$
Figure 4.7. Biaxial shortening of a plane into a developable double-corrugation surface (unfolding of sheet folded according to Miura-ori).
small, the angle $\alpha$ needs to be kept small. Next, consider the deployment of a module consisting of four identical elements like that shown in Fig. 4.8(a), connected by hinges.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure4.8.png}
\caption{Miura-ori module.}
\end{figure}

Define the deployment angle, $\theta$, of this module as the angle between any element and the horizontal, see Fig. 4.8(b): $\theta = 0^\circ$ when the module is fully deployed, i.e. flat, and $\theta = 90^\circ$ when the module is fully packaged, i.e. the four elements overlap. The following expressions can be derived for the ratios between the extensions in the $x$-direction and the $y$-direction, and their respective maximum values

\begin{align*}
\frac{OA}{OA_{max}} &= \cos \left[ \sin^{-1}(\sin \theta \cos \alpha) \right] \\
\frac{OB}{OB_{max}} &= \cos \left[ \frac{\theta}{\cos \left[ \sin^{-1}(\sin \theta \cos \alpha) \right]} \right]
\end{align*} 

A plot of these expressions is shown in Fig. 4.9. For $\alpha = 6^\circ$, which is a good compromise between packaging efficiency and deployment coupling, the first phase of deployment occurs primarily in the $y$-direction, with up to 80% of full expansion being reached in this direction with only 20% in the $x$-direction. In the second deployment phase, expansion is mainly in the $x$-direction.

For $\alpha \to 0$ Miura-ori becomes identical to Fig. 4.5(b), and the plots in Fig. 4.9 tend towards two segments at right angles.

4.3 Simple Folding as a Mechanism

The simple 4-fold or letter-fold system occurs commonly in nature, either singly or in a repetitive structure, and frequently occurs in mechanisms such as leaves or insect wings, when it is actuated from one end (the “base”) only. The 8 different ways in which the letter fold can be arranged are listed in Figure 4.10(b) where the sign of a fold is defined as in Section 4.1. If folds OA and OC are of the same sign, i.e. both hill or valley folds (type 1 in Figure 4.10(b)) then the folding is symmetrical but the mechanism will deploy only if the angles between the folds are not 90°. If folds OA and OC have different signs (type 2) then the system is asymmetrical, folding to left or right depending on the side to which the singleton fold falls. This mechanism will work with any angle between the folds.
The mechanical advantage (MA) of these two types of fold systems is very different. With type 1 the MA is low on opening but the fold can be completely closed again as long as it hasn’t been flattened (i.e. opened fully), when it is impossible to refold. With type 2 the MA is high on opening, but it is difficult to close the fold completely although it can recover from flattening. If the angles at the node are not $90^\circ$ then half the folds are blocked and the system divides itself off into two populations (rows 1, 2, 5 & 6 and 3, 4, 7 & 8 in Figure 4.10(b)).

For calculation of this mechanism, consider the creases as vectors A, C, D and E, all of length 1 and originating at the origin O. From Figure 4.10(a),

\[
\begin{align*}
\mathbf{OA} &= \cos \delta \mathbf{i} + \sin \delta \cos \epsilon \mathbf{j} + \sin \delta \sin \epsilon \mathbf{k} \\
\mathbf{OC} &= \cos \gamma \mathbf{i} + \sin \gamma \mathbf{j} \\
\mathbf{OD} &= d_1 \mathbf{i} + d_2 \mathbf{j} + d_3 \mathbf{k} \\
\mathbf{OE} &= \mathbf{i}
\end{align*}
\]

where $d_1$, $d_2$ and $d_3$ are the components of OD in the $x$, $y$ and $z$ directions; the angles $\alpha$, $\beta$, $\gamma$ and $\delta$ are given, $\epsilon$ is the input variable, and $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the unit vectors in the $x$, $y$ and $z$ directions, respectively. Developing these equations further, and determining $d_1$, $d_2$ and $d_3$ it is possible to calculate the angular movement of $\mathbf{OD}$ in the $x$, $y$ and $z$ planes as the input angle, $\epsilon$, changes (Haas, 1994).

Several of these 4-fold mechanisms, arranged with a variety of angles, can be placed in series giving folding systems of increasing complexity which are commonly found in the folding wings of insects, more especially in the beetles (Coleoptera, lit. sheath-winged) and more especially still in the Staphylinidae or rove-beetles, a family of beetles with very short wing-covers (elytra) which are only a fifth or less of the length of the wings. It is noticeable that the more primitive staphylinids have longer elytra, suggesting that the development of wing folding mechanisms has been important in their evolution. An example of a folding mechanism commonly found in
Julian: The meaning of L, F, R, etc. in the last column of (b) is not explained. Should I delete this column?

Figure 4.10. ??Julian to write.

Julian: have I put $\xi$ in the right place?

Figure 4.11. ??Julian to write.
the Coleoptera is given in Fig. 4.11 which shows three nodes (Haas 1994). If the system is to be completely folded then the sum of all the angles in a triangle must be 180°; the sum of the angles in non-adjacent sectors must also be 180° (Hushimi’s theorem), and \( \xi = \psi + \psi \). This last relationship is derived as follows

\[
\begin{align*}
\alpha + \gamma &= 180^\circ \\
\iota + \phi &= 180^\circ
\end{align*}
\]

This does not change if we add or subtract \( \beta \) or \( \varphi \), so

\[
\begin{align*}
\alpha + \gamma + \beta - \beta &= 180^\circ \\
\iota + \phi + \varphi - \varphi &= 180^\circ
\end{align*}
\]

From Fig. 4.11 it is apparent that

\[
\begin{align*}
\beta + \gamma &= 180^\circ - \xi \\
\varphi + \phi &= 180^\circ - \xi
\end{align*}
\]

so substituting in Equations 4.11-4.12

\[
\begin{align*}
180^\circ - \xi + \alpha - \beta &= 180^\circ \\
180^\circ - \xi + \iota - \varphi &= 180^\circ
\end{align*}
\]

so that

\[
\begin{align*}
\alpha - \beta &= \xi \\
\iota - \varphi &= \xi
\end{align*}
\]

In Fig. 4.11, \( \beta + \xi = \psi \) so that:

\[
\begin{align*}
\beta + \iota - \varphi &= \psi \\
\xi + \varphi + \phi &= 180^\circ
\end{align*}
\]

This means that a straight line, defined by this last equation, must cross the line between points \( O \) and \( C \) such that \( \psi > \beta \) and \( 180^\circ - \psi > 0 \). Thus the angles \( \beta, \iota, \phi \) and \( \varphi \) are interrelated and cannot vary independently. The only unconstrained angles are \( \alpha, \theta, \sigma \) and \( \gamma \).

All this assumes that the unfolded membrane is flat, and that all the angles around a node sum to 180°. This is by no means true in a natural system, where the summed angle can be anywhere between 300° and 400°, giving a variety of over-centre bistable mechanisms which can be used to lock the membrane open or closed.

The other main fold found in nature is the fan in which multiple folds concentrate at a point. This type of fold is found in the wings of the Dermaptera or ear-wigs (often considered a corruption of “ear-wing”, comparing the shape of the wing to that of the human ear). The formation of a fan is only the starting phase in the folding of the wing which is, inevitably, more complex. The fan, once folded, is then folded in half across the fan folds, halving its length. It is then wrapped around with other areas of the wing. The fan is capable of few if any variations since in its simplest
form there are no points where folds cross. It cannot, therefore, be used to generate mechanisms with mechanical advantage or spring mechanisms. An advantage of the fan is that it can be fairly simply deployed or folded.

The linear Miura-ori of the opening beech or hornbeam leaf (Kobayashi et al. 1998), named the ha-ori or leaf-folding, has some of the characteristics of the fan in that the elements are all long and thin and of similar geometry. However, the bases are staggered, whereas the fan is actuated from a point.

4.4 Wrapping Around a Hub

In this section we will consider a different way of packaging a flat, thin membrane. Instead of folding it flat, as in Section 4.2, we will wrap it around a central hub as shown in Fig. 4.12. It can be seen that the folding pattern consists of a symmetric set of hill and valley folds.

![Figure 4.12. Wrapping of a flat membrane.](a) (b)

This packaging scheme was invented in the early 1960’s. Huso (1960) invented a sheet reel for folding compactly the tarpaulin cover of a car. His device consisted of a fixed part, connected to the car roof, and a rotatable hub connected to the tarpaulin: when the hub was rotated the tarpaulin was gradually wound onto it.

The technique was refined by Lanford (1961) who patented the folding apparatus shown in Fig. 4.13, where the regular spacing of hill and valley folds is achieved by means of guiding wires tensioned by weights. This produces a fully-wrapped sheet with a regular saw tooth edge.

More recently, this folding pattern was proposed by Temple and Oswald (Cambridge Consultants, 1989) for the packaging of a solar sail. Their plan was to launch the sail wrapped around
the body of a spacecraft, about 4 m in diameter. Once in orbit, the sail would deploy into a 276 m
diameter disk and would collect enough solar pressure to sail to Mars. For such a large application
it became necessary to work out the folding pattern in more detail.

Guest and Pellegrino (1992) showed that for the abstract case of a membrane of zero thickness
that is wrapped around a prismatic hub with $2n$ sides the $n$ hill folds and $n$ valley folds are straight.
An example is shown in Fig. 4.14(a). This simple pattern can be modified to account for the actual
thickness of the membrane.

To determine the details of this folding pattern we start at the hub vertex, Fig. 4.14(b). Given
the hub angle $\alpha$, we wish to calculate the angles $\beta$, $\gamma$, $\delta$. Obviously,

$$\alpha = (1 - 2/n) \pi$$  \hspace{1cm} (4.21)

and also

$$\alpha + \beta + \gamma + \delta = 2\pi$$  \hspace{1cm} (4.22)

To obtain two more equations we consider the fully wrapped configuration. Because the
membrane has zero thickness, it coincides —in plan view— with the edge of the hub. Hence $BC$, $DE$, $FG$ end up vertical, which implies

$$\overline{ABC} = \delta = \pi/2$$  \hspace{1cm} (4.23)

Because, by symmetry, the angles at vertex $A$ are equal to the corresponding angles at $B$,
Fig. 4.14(b), and $BC$ is vertical after wrapping, we have the following fourth condition, see
Fig. 4.14(c):

$$\gamma - \beta = \pi/2$$  \hspace{1cm} (4.24)

Given Eq. 4.21, the solution of the system consisting of Eqs 4.22-4.24 is
Figure 4.14. Folding pattern for a membrane of zero thickness (from Guest and Pellegrino 1992).
\[ \beta = \frac{\pi}{n}, \quad \gamma = \left( \frac{1}{2} + \frac{1}{n} \right) \pi, \quad \delta = \frac{\pi}{2} \]  

(4.25)

which defines completely the fold pattern in the region next to the hub. Note that \( AC \) bisects the angle between side \( AB \) and the line of side \( OA \).

Next, to define the rest of the folding pattern, we note that the folds \( BC, DE, FG \) end up parallel in Fig. 4.14(c) and hence have to be parallel, since they are coplanar, also in Fig. 4.14(b). They are also equidistant because in Fig. 4.14(c) they pass through adjacent vertices of the hub. With reference to Fig. 4.14(a), this shows that type \( b \) folds are parallel and equidistant.

By symmetry, the same is true for type \( d \) folds.

Finally, we note that \( B, D, F \) and, similarly, \( A, C, E, G \), are collinear because type \( c \) folds pass through the intersections of folds \( b \) and \( d \), in Fig. 4.14(a).

At this point we can draw a complete folding pattern on a flat sheet. First we draw an \( n \)-sided regular polygon representing the hub: its sides are alternate hill and valley folds. Then, we draw \( n \) major fold lines, each forming an angle \( \beta = \pi/n \) with a side of the polygon. Finally, we draw the \( n \) sets of equally spaced, parallel folds \( b \) and \( d \), orthogonal to the sides of the hub.

The fold patterns derived above can be modified to account for a small membrane thickness \( t \). Obviously, the wrapped membrane will no longer coincide with the edge of the hub, a consideration which greatly simplified the analysis for \( t = 0 \). Guest and Pellegrino (1992) assumed that the wrapping of a membrane with \( t \neq 0 \) around an \( n \)-sided polygon is essentially equivalent to the wrapping of a membrane with \( t = 0 \) such that, after folding, its vertices lie on helical curves whose radius increases at a constant rate, based on \( t \).

Thus, they set up a more general version of Eqs. 4.21-4.24, which can be solved numerically for any value of \( t \). The folding pattern for \( n = 6 \), assuming a rather thick membrane (\( t = 2 \text{ mm} \)) is shown in Fig. 4.15 \(^3\). Note that the effect of non-zero thickness is to bend the major fold lines.

Finally, a circular hub can be seen as a limiting case for \( n \to \infty \). However, as the number of major hill and valley folds cannot be very large, what happens in practice is that the folding pattern is essentially determined by the actual value \( n \). Near the hub, localised stretching and wrinkling of the membrane account for the different shape between the required, prismatic hub shape and circular hub.

### 4.5 References


\(^3\) The reader is invited to enlarge this pattern on a photocopier, so that the 50 mm line has approximately this length, and to make a model. Because the thickness, \( t \), assumed in the calculation of the folding pattern is much greater than the thickness of the paper, the sheet will not package very tightly.
Figure 4.15. Folding pattern for a thick membrane.


